Crossian Resonance

A Multidisciplinary Research Journal **ISSN 0976-5417**
(A refereed Biannual Published in June and December)
Vol. 15, No.2, December 2024
https://research.holycrossngl.edu.in/

The Chromatic Restrained Domination Number on Line Graph of Standard Graphs

Divinelin Kumari R. and *Angel Jebitha M.K.
Department of Mathematics, Holy Cross College (Autonomous), Nagercoil – 629 004
Affiliated to Manonmaniam Sundaranar University, Tirunelveli – 627 012

ABSTRACT

Let $G = (V, E)$ be a graph and $L(G)$ be the line graph of G. A subset D of $V(G)$ is said to be a chromatic restrained dominating set (or crd-set) of G if D is a restrained dominating set of G and $\chi(< D >) = \chi(G)$. The minimum cardinality taken over all minimal chromatic restrained dominating sets is called the chromatic restrained domination number of G and is denoted by $\gamma_r^c(G)$. In this paper, we compute the chromatic restrained domination number for the line graph of some standard graphs.

Keywords: *Domination, Restrained Domination, Chromatic Number, Line Graphs*
AMS Subject Classification: *05C15, 05C69*

1. Introduction

All the graphs $G = (V, E) = (n, m)$ considered here are simple, finite and undirected, with neither loops nor multiple edges. For $D \subseteq V$, the subgraph induced by D is denoted by $\langle D \rangle$. The *degree* of a vertex v in a graph G, denoted by $\deg(v)$ is the number of edges incident with v. A $k-$vertex coloring of a graph, or simply a $k-$coloring, is an assignment of $k-$colors to its vertices. The coloring is proper if no two adjacent vertices are assigned the same color. A coloring in which $k-$ colors are used is a $k-$coloring. A graph is $k-$colorable if it has a proper $k-$coloring. The minimum k for which a graph G is $k-$colorable is called its *chromatic number* and denoted by $\chi(G)$. Graph Theory terminologies which are not defined here can be seen in [1] and [2].

A set $D \subseteq V$ of vertices in a graph G is called a dominating set if every vertex $u \in V$ is either an element of D or is adjacent to an element of D. The minimum cardinality taken over all minimal dominating sets is called the *domination number* of G and is denoted by $\gamma(G)$. A set $D \subseteq V$ is a restrained dominating set if every vertex in $V - D$ is adjacent to a vertex in D and another vertex in $V - D$ [3]. The minimal cardinality taken over all minimal restrained dominating sets is called the *restrained domination number* of G and is denoted by $\gamma_r(G)$. A set D is a γ_r-set if D is a restrained dominating set of cardinalities $\gamma_r(G)$.

Corresponding Author: ***angeljebitha@holycrossngl.edu.in***

For a graph G with edges, the *line graph* $L(G)$ is the graph whose vertices correspond to the edges (lines) of G, and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent (that is, are incident with a common vertex). In 2018, E. Murugan and J. Paulraj Joseph determined the bounds on the sum of the domination number of a graph and its line graph. They also characterised upper and lower bounds on the domination number of line graphs [4]. The *bistar graph* $B_{r,s}$ is the graph obtained by joining the center vertices of two-star graphs $K_{1,r}$ and $K_{1,s}, r, s \geq 2$.

A set $D \subseteq V$ is a *chromatic preserving set* or a cp-set if $\chi(< D >) = \chi(G)$ and the minimum cardinality taken over all cp-set in G is called the *chromatic preserving number* or cp-number of G, denoted by $\text{cpn}(G)$. This new concept was introduced by T. N. Janakiraman and M. Poobalaranjani [5]. They also defined the concept of a dom-chromatic set of a graph. A subset D of V is said to be a *dom-chromatic set* (or dc-set) if D is a dominating set and $\chi(< D >) = \chi(G)$. The minimum cardinality taken over all minimal dom-chromatic sets in G is called the *dom-chromatic number* and is denoted by $\gamma_{ch}(G)$. In [6], they determined dom-chromatic numbers for several classes of graphs and established key results in this area. Based on this, S. Balamurugan et al [7], [8], [9], [10] introduced and studied the concepts of chromatic strong domination, chromatic total domination, chromatic connected domination, chromatic weak domination and so on. Additionally, J. Joseline Manora et al [11], [12] explored connected majority dom-chromatic number of a graph and majority dom-chromatic set of a graph. Similar to these works, several authors have explored various types of dom-chromatic sets. In this paper, we study a new domination parameter, the chromatic restrained domination number of line graphs.

2. Chromatic Restrained Domination Number for Line Graphs

In this section, we obtained the chromatic restrained domination number for the line graph of some standard graphs.

Definition 2.1 Let $G = (V, E)$ be a graph and $L(G)$ be the line graph of G. A subset D of V is said to be a chromatic restrained dominating set (or crd-set) if D is a restrained dominating set and $\chi(< D >) = \chi(G)$. The minimum cardinality taken over all minimal chromatic restrained dominating sets is called chromatic restrained domination number and is denoted by $\gamma_r^c(G)$. Throughout this paper, we denote the chromatic restrained domination number of line graphs by $\gamma_r^c(L(G))$.

Theorem 2.2 Let $G = K_n$ be the complete graph on n vertices. Then

$$\gamma_r^c(L(K_n)) = \begin{cases} \dfrac{n(n-1)}{2} - 2 & if\ n\ is\ odd, n \geq 5 \\ n - 1 & if\ n\ is\ even, n \geq 4 \end{cases}$$

Proof: Let $V(K_n) = \{v_1, v_2, v_3, \dots, v_n\}$ and $E(K_n) = \{m_{ij}/1 \leq i \leq n, 1 \leq j \leq n, m_{ij} = m_{ji,} i \neq j\}$. Then $V\big(L(K_n)\big) = E(K_n)$ with cardinality $\frac{n(n-1)}{2}$. Also $\chi(K_n) = n$.

Case (i): n is odd

Then $\chi\big(L(K_n)\big) = n$. Consider $D = \{m_{i(i+1)}/1 \leq i \leq n-2, i \text{ is odd}\}$ and D is independent. Then D is a restrained dominating set of $L(K_n)$ with cardinality $\left\lfloor \frac{n}{2} \right\rfloor$. Since any minimal restrained dominating set is independent, $\chi(\langle D \rangle) = 1 \neq \chi\big(L(K_n)\big)$. Therefore, D is not a chromatic restrained dominating set. Let $S = V\big(L(K_n)\big) - \{m_{12}, m_{13}\}$. Then S is a restrained dominating set and $\chi(\langle S \rangle) = n = \chi\big(L(K_n)\big)$. Therefore, S is a chromatic restrained dominating set with cardinality $\frac{n(n-1)}{2} - 2$ and $\gamma_r^c\big(L(K_n)\big) \leq \frac{n(n-1)}{2} - 2$. We show that S is a minimal chromatic restrained dominating set of $L(K_n)$. Suppose not, then there exists a chromatic restrained dominating set S' such that $S' \subset S$. Then there exists a vertex $x \in S$ and $x \notin S'$. Thus $\chi(\langle S' \rangle) < \chi\big(L(K_n)\big)$, which is a contradiction. Therefore, S is a minimal chromatic restrained dominating set of $L(K_n)$, so that $\gamma_r^c\big(L(K_n)\big) \geq \frac{n(n-1)}{2} - 2$. Hence, $\gamma_r^c\big(L(K_n)\big) = \frac{n(n-1)}{2} - 2$, where n is odd.

Case (ii): n is even

Then $\chi\big(L(K_n)\big) = n - 1$. Let $D = \{m_{i(i+1)}/1 \leq i \leq n, i \text{ is even}\}$. Then D is a restrained dominating set of $L(K_n)$ with cardinality $\frac{n}{2}$. Since D is independent, $\chi(\langle D \rangle) = 1 \neq \chi\big(L(K_n)\big)$. Therefore, D is not a chromatic restrained dominating set of $L(K_n)$. Consider $D_i = \{m_{ij}/1 \leq j \leq n, i \neq j\}$. Then $D_i'^s$ are the subsets of $V\big(L(K_n)\big)$ with same cardinality and $\chi(\langle D_i \rangle) = n - 1$. Also $D_i'^s$ are restrained dominating sets of $L(K_n)$. Since $\langle D_i \rangle$ is complete, then for any $D_i - \{m_{ij}\}, 1 \leq i \leq n$, the chromatic number is less than $n - 1$. Therefore, $D_1, D_2, \dots, D_n$ are the only sets of $L(K_n)$ with chromatic number $n - 1$ and minimum cardinality. Therefore $D_i'^s$ are the chromatic restrained dominating sets of $L(K_n)$ and $\gamma_r^c\big(L(K_n)\big) = n - 1$.

Observation 2.3: $\gamma_r^c\big(L(K_n)\big) = \begin{cases} 1 & if\ n = 2 \\ 3 & if\ n = 3 \end{cases}$

Observation 2.4:

(1) For any path P_n,

$$\gamma_r^c\big(L(P_n)\big) = \gamma_r^c(P_{n-1}) = \begin{cases} \frac{n-1}{3} + 2 & if\ n \equiv 1 (mod\ 3) \\ \left\lceil \frac{n-1}{3} \right\rceil + 2 & if\ n \equiv 2 (mod\ 3) \\ \left\lceil \frac{n-1}{3} \right\rceil + 1 & if\ n \equiv 0 (mod\ 3) \end{cases}$$

(2) For any cycle C_n,

(i) if n is odd, then $\gamma_r^c\big(L(C_n)\big) = \gamma_r^c(C_n) = n$

(ii) if n is even, then

$$\gamma_r^c\big(L(C_n)\big) = \gamma_r^c(C_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil & if \ \ n \equiv 1(mod\ 3) \\ \left\lceil \frac{n}{3} \right\rceil + 1 & if \ \ n \equiv 2(mod\ 3) \\ \frac{n}{3} + 2 & if \ n \equiv 0(mod\ 3) \end{cases}$$

Theorem 2.5: For $n \geq 3$, $\gamma_r^c\left(L\big(K_{1,n-1}\big)\right) = n - 1$.

Proof: Let $V\big(K_{1,n-1}\big) = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$ where $\deg(v_0) = n - 1$ and $v_1, v_2, \ldots, v_{n-1}$ are the vertices of degree one. Then $E\big(K_{1,n-1}\big) = \big\{m_i/1 \leq i \leq n-1, m_i = v_0v_i \in E\big(K_{1,n-1}\big)\big\} = V\left(L\big(K_{1,n-1}\big)\right)$. Also $\chi\left(L\big(K_{1,n-1}\big)\right) = n - 1$. Any singleton set D is a restrained dominating set of $L\big(K_{1,n-1}\big), n > 3$. Clearly, $\chi(\langle D\rangle) = 1 \neq \chi\left(L\big(K_{1,n-1}\big)\right)$. Therefore, D is not a chromatic restrained dominating set of $L\big(K_{1,n-1}\big)$. Since $\chi\left(L\big(K_{1,n-1}\big)\right) = n - 1, S = \{m_1, m_2, \ldots, m_{n-1}\}$ is the minimal set with respect to the property that the induced subgraph has chromatic number $n - 1$, as $\langle S\rangle$ is a complete graph on $n - 1$ vertices. Also S is a restrained dominating set of $L\big(K_{1,n-1}\big)$. Therefore, S is a chromatic restrained dominating set of $L(K_{1,n-1})$ with cardinality $|S| = \left|V(L\big(K_{1,n-1}\big))\right| = n - 1$. Therefore, $\gamma_r^c\left(L\big(K_{1,n-1}\big)\right) = n - 1$.

Theorem 2.6: For $n \geq 4, \gamma_r^c\big(L(W_n)\big) = n - 1$.

Proof: Let $V(W_n) = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$ with $n - 1$ outer vertices and $E(W_n) = \big\{m_i, m_{j(j+1)}, m_{1(n-1)}/m_i = v_0v_i, 1 \leq i \leq n-1, 1 \leq j \leq n-2\big\} = V\big(L(W_n)\big)$. Also, $\deg(v_0) = n - 1$ and $\chi\big(L(W_n)\big) = n - 1$. Since v_0 is a vertex of maximum degree in W_n, the induced subgraph of $L(W_n)$ formed from the edges incident with v_0 has chromatic number $n - 1$. Also $D = \{m_i/1 \leq i \leq n-1\}$ is a restrained dominating set of $L(W_n)$ with cardinality $|D| = n - 1$. Therefore, D is the only chromatic restrained dominating set of $L(W_n)$ with minimum cardinality. Hence. $\gamma_r^c\big(L(W_n)\big) = n - 1$.

Theorem 2.7: For any complete bipartite graph $K_{r,s}, \gamma_r^c\left(L\big(K_{r,s}\big)\right) = \max\{r, s\}$.

Proof: Let $V\big(K_{r,s}\big) = V_1 \cup V_2$ where $V_1 = \{v_1, v_2, \ldots, v_r\}$ and $V_2 = \{u_1, u_2, \ldots, u_s\}$. Also $E\big(K_{r,s}\big) = \{m_{ij}/m_{ij} = v_iu_j, 1 \leq i \leq r, 1 \leq j \leq s, m_{ij} \neq m_{ji}\} = V\left(L\big(K_{r,s}\big)\right)$ and

$\chi\left(L(K_{r,s})\right) = \max\{r, s\}$. Let $r \leq s$. Any set D containing all the edges incident with any vertex in V_2 of $K_{r,s}$ is a restrained dominating set of $L(K_{r,s})$ with $|D| = r$ and $\gamma_r\left(L(K_{r,s})\right) \leq r$. Since $\gamma\left(L(K_{r,s})\right) = r, \gamma_r\left(L(K_{r,s})\right) \geq r$.Therefore, $\gamma_r\left(L(K_{r,s})\right) = r$. Since $\langle D\rangle$ is a complete graph on r vertices, $\chi(\langle D\rangle) = r$. If $r = s$, then $\chi(\langle D\rangle) = r = \chi\left(L(K_{r,s})\right)$. Therefore, D is a chromatic restrained dominating set of $L(K_{r,s})$. Let $r \neq s$. Then $\chi(\langle D\rangle) \neq \chi\left(L(K_{r,s})\right)$ and D is not a chromatic restrained dominating set of $L(K_{r,s})$. Consider $S = \{m_{11}, m_{12}, m_{13}, \ldots, m_{1s}\}$ where $m_{11}, m_{12}, m_{13}, \ldots, m_{1s}$ are the edges incident with $v_1 \in V_1$. Then S is a restrained dominating set of $L(K_{r,s})$ and $\chi(\langle S\rangle) = s = \max\{r, s\} = \chi\left(L(K_{r,s})\right)$, since $\langle S\rangle$ is a complete graph on s vertices. Therefore, S is a chromatic restrained dominating set of $L(K_{r,s})$ and $\gamma_r^c\left(L(K_{r,s})\right) \leq |S| = s$. Since $\chi\left(L(K_{r,s})\right) = s, \gamma_r^c\left(L(K_{r,s})\right) \geq s$. Therefore, $\gamma_r^c\left(L(K_{r,s})\right) = s = \max\{r, s\}$.

Theorem 2.8: For any bistar graph $B_{r,s}, \gamma_r^c\left(L(B_{r,s})\right) = \max\{r, s\} + 1$.

Proof: Let $V(B_{r,s}) = \{v_0, u_0, v_1, v_2, \ldots, v_r, u_{r+1}, u_{r+2}, \ldots, u_{r+s}\} = \{v_0, u_0, v_i, u_j / 1 \leq i \leq r, r+1 \leq j \leq r+s\}$ where v_0 and u_0 are the center vertices of $K_{1,r}$ and $K_{1,s}$ and $|V(B_{r,s})| = r + s + 2$. Then $E(B_{r,s}) = \{m_0, m_i, m_j / m_0 = v_0u_0, m_i = v_0v_i, m_j = u_0u_j, 1 \leq i \leq r, r + 1 \leq j \leq r+s\} = V\left(L(B_{r,s})\right)$. Let u_0 be a vertex of maximum degree in $B_{r,s}$ and $\Delta(B_{r,s}) = s + 1$. Also $\chi\left(L(B_{r,s})\right) = s + 1$. Consider $D = \{m_0\}$. Then D is a restrained dominating set of $L(B_{r,s})$ and $\gamma_r\left(L(B_{r,s})\right) = 1$. Also $\chi(\langle D\rangle) = 1 \neq \chi\left(L(B_{r,s})\right)$. Therefore, D is not a chromatic restrained dominating set of $L(B_{r,s})$. Let $S = \{m_0, m_{r+1}, m_{r+2}, \ldots, m_{r+s}\}$. Clearly, S is a restrained dominating set of $L(B_{r,s})$. Since $\langle S\rangle$ is a complete graph on $s + 1$ vertices, $\chi(\langle S\rangle) = s + 1 = \chi\left(L(B_{r,s})\right)$. Therefore, S is a chromatic restrained dominating set of $L(B_{r,s})$ and $\gamma_r^c\left(L(B_{r,s})\right) \leq s + 1$. Since $\{m_0, m_{r+1}, m_{r+2}, \ldots, m_{r+s}\}$ is the minimal set with respect to the property that the induced subgraph has chromatic number $s + 1$, $\gamma_r^c\left(L(B_{r,s})\right) \geq s + 1$. Therefore, $\gamma_r^c\left(L(B_{r,s})\right) = s + 1 = \max\{r, s\} + 1$.

Theorem 2.9: For any friendship graph F_m,$m \geq 2$, $\gamma_r^c\left(L(F_m)\right) = 3m$.

Proof: Let $|V(F_m)| = 2m + 1$ and $V(F_m) = \{v_0, v_1, v_2, v_3, v_4, \ldots, v_{2m-1}, v_{2m}\}$ where v_0 is the vertex of maximum degree in F_m. Then $E(F_m) = \{m_i, m_{j(j+1)} / 1 \leq i \leq 2m, 1 \leq j \leq 2m - 1, j \text{ is odd}\} = V(L(F_m))$. Also, $\chi\left(L(F_m)\right) = 2m$. Let $D =$

$\{m_{12}, m_{34}, m_{56}, m_{78}, \ldots, m_{(2m-1)(2m)}\}$. Then D is a restrained dominating set of $L(F_m)$ and cardinality of D is m. Therefore, $\gamma_r(L(F_m)) \leq m$. Since $\gamma(L(F_m)) = m, \gamma_r(L(F_m)) \geq m$. Therefore, $\gamma_r(L(F_m)) = m$. Since D is independent, $\chi(\langle D \rangle) = 1 \neq 2m$. Therefore, D is not a chromatic restrained dominating set of $L(F_m)$. Consider $D_1 = \{m_i / 1 \leq i \leq 2m\}$. Then D_1 is the minimal set whose induced subgraph has chromatic number $2m$. But D_1 is not a restrained dominating set of $L(F_m)$, since $V(L(F_m)) \backslash D_1$ is an independent set. Therefore, D_1 is not a chromatic restrained dominating set of $L(F_m)$. Thus, for any proper subset $S = V(L(F_m)) - \{m_i\}$ and $V(L(F_m)) - \{m_{j(j+1)}\}$, either $\chi(\langle S \rangle) < 2m$ or S is not a restrained dominating set. Therefore, $V(L(F_m))$ is the only chromatic restrained dominating set of $L(F_m)$ and $\gamma_r^c(L(F_m)) = |V(L(F_m))| = 3m$.

Observation 2.10: For any line graph $L(G)$ of G, $1 \leq \gamma_r^c(L(G)) \leq \frac{n(n-1)}{2} - 2$.

Theorem 2.11: If G is a graph, then $\gamma_r^c(L(G)) = 1$ if and only if $G = K_2$.

Proof: Assume that $\gamma_r^c(L(G)) = 1$. Then $\chi(L(G)) = 1$ and $L(G) = nK_1$ is the only possible line graph with $\chi(L(G)) = 1$. But $\gamma_r^c(nK_1) = n$, for $n \geq 1$.Therefore, $L(G) = K_1$. Then, G is a graph with an edge, which is K_2. Therefore, $G = K_2$. The converse is trivial.

3. Conclusion

In this paper, we have determined the chromatic restrained domination number for the line graph of certain standard graphs and observed that $1 \leq \gamma_r^c(L(G)) \leq \frac{n(n-1)}{2} - 2$. An encouraging direction for future research is to analyse the extremal graphs that represent the upper limits of the chromatic restrained domination number in line graphs.

References

1. Bondy J. A., Murty U. S. R. Graph Theory with Applications. Springer; 2008.
2. Teresa W. Haynes., Stephen T. Hedetniemi., Peter J. Slater. Fundamentals of Domination in Graphs. Marcel Dekker; 1998.
3. Gayla S. Domke., Johannes H. Hattingh., Stephen T. Hedetniemi., Renu C. Laskar., Lisa R. Markus. Restrained Domination in Graphs. Discrete Mathematics. 1999; 203: 61 - 69.
4. Murugan E., Paulraj Joseph. On the domination number of a graph and its line graph. International Journal of Mathematical Combinatorics. 2018; 170 - 181.
5. Janakiraman T. N., Poobalaranjani M. On the Chromatic Preserving Sets. International Journal of Engineering Science, Advanced Computing and Bio-Technology. 2010; 1: 29 - 42.

6. Janakiraman T. N., Poobalaranjani M. Dom-Chromatic sets of graphs. International Journal of Engineering Science, Advanced Computing and Bio-Technology. 2011; 2: 88 - 103.
7. Balamurugan S., Prabakaran G., Swaminathan V. On Chromatic Strong Dominating Sets in Graphs. International Journal of Engineering Science, Advanced Computing and Bio-Technology. 2011; 2: 139 - 149.
8. Balamurugan S., Anitha M., Angala Eswari M., Kalaiselvi S. Chromatic total domination in graphs. Journal of Discrete Mathematical Sciences and Cryptography. 2019; 22: 745 - 751.
9. Balamurugan S., Anitha M., Kalaiselvi S. Chromatic connected domination in graphs. Journal of Discrete Mathematical Sciences and Cryptography. 2019; 22: 753 - 760.
10. Selvalakshmi P., Balamurugan S. A Note on Chromatic Weak Dominating Sets in Graphs. International Journal of Mathematics Trends and Technology (IJMTT). 2018; 53: 464 - 475.
11. Joseline Manora J., Mekala R. Majority dom-chromatic set of a graph. Bulletin of Pure and Applied Sciences. 2019; 38 E (Math & Stat.): 289 - 296.
12. Joseline Manora J., Mekala R. Connected majority dom-chromatic number of a graph. Advances and Applications in Mathematical Sciences. 2022; 21: 1937 - 1950.